Mathematics Learning Centre

The University of Sydney

Counting Techniques

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Contents

1 Introduction

Calculations in probability theory often involve working out the number of different ways in which something can happen. Since simply listing the ways can be very tedious (and often unreliable), it is helpful to work out some techniques for doing this kind of counting.

1.1 How to use this book

You will not gain much by just reading this book. Have pencil and paper ready to try to work through the examples before reading their solutions. Do all the exercises. It is important that you try hard to complete the exercises on your own, rather than refer to the solutions as soon as you are stuck.

1.2 Assumed knowledge

You will need to use a calculator for the exercises. Cancel manually as much as possible to avoid multiplying a large series of numbers by calculator.

For example, evaluate

$$\frac{30.29.28.27.26.25.24.23}{8.7.6.5.4.3.2.1}.$$

This may be reduced to

$$\frac{\cancel{30}.29.\cancel{28}.27.\overset{13}{\cancel{26}}.25.\cancel{24}.23}{\cancel{8}.\cancel{7}.\cancel{6}.\cancel{5}.\cancel{4}.\cancel{3}.\cancel{2}.1} = 29.27.13.25.23 = 5852925.$$

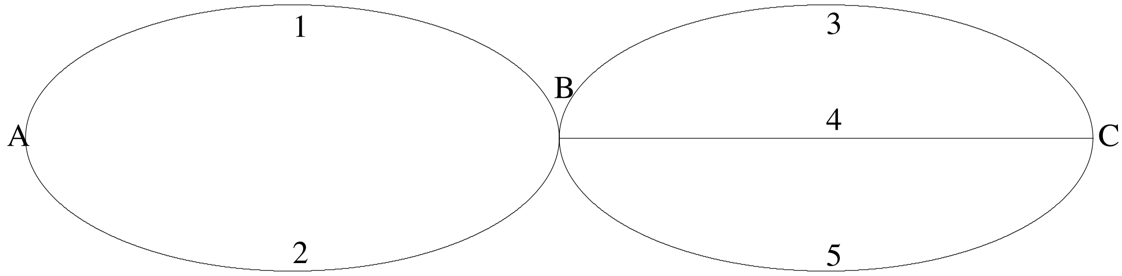
1.3 Objectives

By the time you have worked through this unit you should

- be able to calculate the number of ways in which an event can take place when the event has two or more stages, each having a number of choices,
- be familiar with factorial notation,
- know the difference between permutations and combinations and be able to solve problems involving these,
- understand the concept of binomial coefficients and be able to calculate them.

2 A basic counting principle

Suppose there are three towns A, B and C, with 2 roads from A to B and 3 roads from B to C, as shown in the diagram.



Let's count the number of different routes we can take in travelling from A to C. This will be easier to do if we number the roads as shown above.

We have two choices of route from A to B. Whichever of these we choose, when we reach B we have three choices of route from B to C. So the total number of routes from A to C is 2×3. We can list them:

1,3	1,4	1,5
2,3	2,4	2,5

Now let's try to find the number of routes from A to C if there are 3 roads from A to B and 4 from B to C. (Try drawing a diagram, number the roads, and then listing the routes systematically.)

Solution

There are 3 choices of route from A to B. For each of these, there are 4 choices from B to C. So there are $3 \times 4 = 12$ choices of route from A to C.

Finally, suppose there are 3 roads from A to B, 4 roads from B to C and 3 roads from C to another town, D. How many possible routes are there?

Solution

As above, there are 12 choices of route from A to C. For each of these, there are 3 choices from C to D. So there are 12×3 (ie $3 \times 4 \times 3$) choices of route from A to D. (If you are at all unsure about this, draw a diagram, number the different routes, and list all the routes in a systematic way.)

These examples illustrate the basic counting principle which we can express informally as:

> **To find the number of ways of doing something, multiply the number of choices available at each stage.**

2.1 Exercises

1. The local newsagent has gift wrapping paper and ribbon on sale. The paper comes in green, blue, purple and white, and the ribbon in red and gold. If you buy one package of paper and one of ribbon, how many different colour combinations can you choose?

2. Sam is buying fruit trees for his garden. He plans to plant one peach, one apple, one plum and one cherry tree. The nursery recommends two varieties of peach, four of apple, six of plum and three of cherry for his area. How many possible different groups of trees could he plant? If Sam has already decided on the cherry, how many choices are left for the other trees?

3. A New South Wales netball team plays teams from the other states in the competition so that they play four matches each month. Each match has three possible results for the NSW team — win, lose or draw. How many different sequences of results are possible in a month for NSW.

3 Factorial notation

Many counting problems involve multiplying together long strings of numbers. Factorial notation is simply a short hand way of writing down some of these products.

The symbol $n!$ reads as 'n factorial' and means $n(n-1)(n-2)\cdots 2.1$.

For example

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 1! &= 1 \end{aligned}$$

We also define 0! to be 1.

Note: Many people think that 0! ought to be 0, but this would give rise to problems of dividing by 0. We shall see that the formulae we'll be deriving make more sense if $0! = 1$.

3.1 Exercises

1. Evaluate

a. $\dfrac{7!}{5!\,3!}$

b. $\dfrac{15!}{9!\,6!}$.

2. Use a calculator to evaluate

a. 9!

b. 32! (What does the calculator representation of this very large number mean?)

3. What is the largest factorial your calculator will evaluate?

4. Check your answers to Question 1. on your calculator.

5. Write out in full

a. $m!$

b. $\dfrac{m!}{r!}$

c. $(m-r)!$

d. $\dfrac{m!}{r!(m-r)!}$

4 Permutations

The word ‘permutations’ means ‘arrangements’. We use it to refer to the number of ways of arranging a set of objects. In other words, we use permutations when we are concerned about ‘order’.

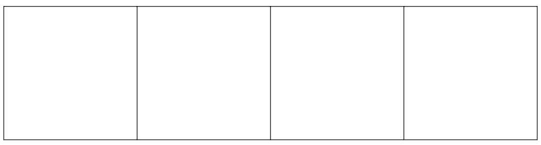
Try to work out each of the following examples for yourself before reading the solutions.

Example

How many different 4 letter arrangements can we make of the letters in the word ‘cats’, using each letter once only.

Solution

We have four positions to fill.



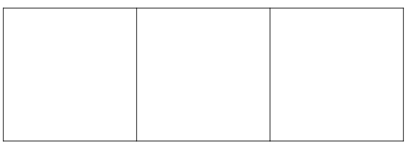
There are four choices for position 1. For each of those choices there are 3 letters left, and so 3 ways to fill position 2. So by the counting principle there are 4×3 ways of filling the first 2 positions. For each of these choices there are now 2 letters left and there are two ways of filling the third position. The remaining letter must then go in the last position.

Thus by the counting principle, there are $4 \times 3 \times 2 \times 1 = 4!$ possible arrangements, ie 24 of them. Try writing them out as a check.

Example

How many ways can the numbers 7, 8 and 9 be arranged using each number once? Write out all the permutations of 7, 8 and 9 to check that your answer is correct.

Solution



There are 3 places to fill. This can be done in $3 \times 2 \times 1 = 3! = 6$ ways.

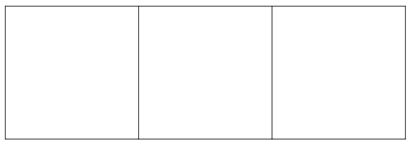
They are

7, 8, 9	7, 9, 8
8, 7, 9	8, 9, 7
9, 7, 8	9, 8, 7.

Example

How many 3 letter 'words' can be made using the letters a, b, c, d, e, and f if each letter can be used at most once?

Solution



The first box can be filled in 6 ways, the second in 5 ways and the third in 4 ways, ie the number of permutations of 6 letters taken 3 at a time is $6 \times 5 \times 4$.

If we wish the use the convenient factorial notation, multiply top and bottom by $3 \times 2 \times 1$ to get

$$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{6!}{3!}.$$

Example

How many 3 digit numbers can be made from the digits 1, 2, $\cdots$, 9, if each can be used once? How may 7 digit numbers can be made? Express your answers in factorial notation.

Solution

The number of 3 digit numbers is $9 \times 8 \times 7 = \dfrac{9!}{6!}$.

The number of 7 digit numbers is $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = \dfrac{9!}{2!}$.

Now let's work out a general formula for the number of arrangements of n *different* objects taken r at a time.

Here there are r boxes to fill.

The first can be filled in n ways.

The second can be filled in $(n-1)$ ways.

The third can be filled in $(n-2)$ ways.

$\ldots$

The rth box can be filled in $n-(r-1) = (n-r+1)$ ways.

So the total number of arrangements is

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots 2.1}{(n-r)(n-r-1)\cdots 2.1} = \frac{n!}{(n-r)!}.$$

Here we multiplied top and bottom by $(n-r)(n-r-1)\cdots 2.1$.

We denote this as ${}^n\mathrm{P}_r$.

4.1 Exercises

1. Write in factorial notation $^{20}\mathrm{P}_5$, $^{7}\mathrm{P}_2$.

2. Evaluate $^{6}\mathrm{P}_4$, $^{6}\mathrm{P}_2$.

3. How many different signals can be sent by displaying three flags on a mast if there are six different flags available?

4. Make up two questions with the answer $^{7}\mathrm{P}_2$.

5. Does your calculator have a key $^{n}\mathrm{P}_r$? If so use it to check your answers to Question 2.

6. Write in factorial notation:

a. $^{m}\mathrm{P}_r$

b. $^{n}\mathrm{P}_k$

c. $^{n}\mathrm{P}_{n-k}$.

5 Combinations

Combinations is a technical term meaning 'selections'. We use it to refer to the number of different sets of a certain size that can be selected from a larger collection of objects where **order does not matter**.

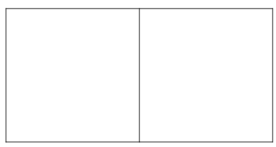
Example

How many committees of two people can be chosen from this group of four people:

Jones, Chan, Vello, Harris?

Solution

There are two committee members to be chosen so we have two places to fill.



There are 4 ways of choosing the first member and 3 ways of choosing the second. Thus it would seem at a first glance that there are 12 ways to choose the committee — let us list them:

Chan, Jones	Jones, Harris	Harris, Vello	Vello, Chan
Chan, Harris	Jones, Vello	Harris, Chan	Vello, Jones
Chan, Vello	Jones, Chan	Harris, Jones	Vello, Harris.

However, when we look more closely at this list we notice that only 6 of these form different committees. Jones and Harris, for example, make up the same committee as Harris and Jones. The order in which people are chosen does not matter. To obtain the correct answer we cannot just find

$$^4P_2 = \frac{4!}{2!} = 4 \times 3 = 12$$

but must also divide by the number of ways the committee can be arranged, ie 2!, giving the answer 6.

Example

How many distinct sets of 3 differently coloured scarves can be bought if the shop has scarves in 8 different colours?

Solution

There are 8 different colours of scarves available, so the first scarf can be chosen in any one of these 8 colours, that is, in 8 different ways. Since the scarves selected are all to be in different colours, once the first scarf is chosen, there are 7 ways of choosing the colour of the second scarf, and corresponding to each of these, 6 ways of choosing the third. Thus it appears there are $8 \times 7 \times 6$ ways of choosing scarves in 3 different colours, ie $\frac{8!}{5!}$.

However these sets of 3 will not all be distinct. Suppose that a red scarf is selected first, then a blue and then a white. This will result in the same set of scarves as if a white were chosen first, then a blue and then a red. In fact, since a set of 3 colours can be arranged in $3 \times 2 \times 1 = 3!$ different ways, there will be 3! different choices which will result in the same set of scarves. Hence we must divide the number obtained by 3!. So the answer to our problem is

$$\frac{8!}{3!\,5!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1}.$$

Now, once again, let's work out a general formula. This time we want the number of ways of choosing r objects from n distinct objects when the order in which the objects are chosen does not matter. Any two selections containing the same r objects are considered to be the same.

We will denote this by ^nC_r or $\begin{pmatrix} n \\ r \end{pmatrix}$ read as 'n choose r'.

The number of ways of choosing an **ordered** set of r objects out of n is equal to the number of ways of choosing an **unordered** set of r objects out of n **times** the number of ways of arranging r objects in order.

That is,

$$\begin{aligned} ^nP_r &= {}^nC_r \times r! \\ \text{no. of ordered selections} &= \text{no. of unordered selections} \times \text{no. of ways of arranging them.} \end{aligned}$$

Since we already know that

$$^nP_r = \frac{n!}{(n-r)!}\ ,$$

this tells us that

$$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 2.1}.$$

(Note: there are r factors in both the top and bottom of this fraction.)

Going back to the previous two examples, we can now write them in this notation.

The number of committees equals

$$^4C_2 = \frac{4!}{2!\ 2!} = \frac{4 \times 3}{2 \times 1} = 6.$$

The number of sets of scarves equals

$$^8C_3 = \frac{8!}{5!\ 3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56.$$

5.1 Exercises

1. Write out in factorial notation and hence evaluate $^7C_3$, $^7C_4$, $^7C_0$.

2. Evaluate $\frac{^9C_5}{^9C_6}$.

3. Fill in the numerators:

a. $^tC_4 = \frac{}{4.3.2.1}$

b. $^7C_t = \frac{}{t(t-1)\cdots 1}$.

4. Find the key for ^nC_r on your calculator. Use it to check your answers to questions 1 and 2.

5. Express in terms of factorials: ^nC_r and $^nC_{n-r}$. What do you notice?

6. In how many ways can twelve pieces of fruit be divided into two baskets containing five and seven pieces of fruit respectively?

7. Make up two questions with answer $^7C_2$.

6 Binomial coefficients

Let's consider the expansion of $(x+a)^n$ for $n = 0, 1, 2, 3, 4 \cdots$.

$$\begin{aligned}
(x+a)^0 &= 1 \\
(x+a)^1 &= x+a \\
(x+a)^2 &= x^2+2ax+a^2 \\
(x+a)^3 &= x^3+3x^2a+3xa^2+a^3 \\
(x+a)^4 &= x^4+4x^3a+6x^2a^2+4xa^3+a^4 \\
(x+a)^5 &= x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5
\end{aligned}$$

We notice the following properties.

1. For each term, the exponents of x and a add up to n. For example, in the expansion $(x+a)^5$, the powers of x and a always add to 5.

2. If we write dowm just the coefficient of each term in the expansion above , we obtain the triangle known as Pascal's triangle.

$$\begin{array}{lccccccccccc}
(x+a)^0 & & & & & & 1 & & & & & \\
(x+a)^1 & & & & & 1 & & 1 & & & & \\
(x+a)^2 & & & & 1 & & 2 & & 1 & & & \\
(x+a)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(x+a)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(x+a)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

and so on.

It can be seen that each coefficient in the triangle can be obtained by adding the two numbers directly above it, eg $1+4=5$, $6+4=10$.

Let's look more closely at how these coefficients are obtained. Consider

$$\begin{aligned}
(x+a)^2 &= (x+a)(x+a) \\
&= x^2+xa+ax+a^2.
\end{aligned}$$

Notice that there are two ways of obtaining a term in x: by choosing 'x' from the first bracket and 'a' from the second, and by choosing 'a' from the first bracket and 'x' from the second. Hence the coefficient of ax in the expansion of $(x+a)^2$ is ${}^2C_1 = 2$, the number of ways of choosing one 'x' from two brackets.

Now look at

$$\begin{aligned}
(x+a)^3 &= (x+a)(x+a)(x+a) \\
&= x^3+x^2a+xax+xa^2+ax^2+axa+a^2x+a^3.
\end{aligned}$$

We see that the terms involving x^2 are: x^2a, xax and ax^2 which when added together give $3x^2a$.

We can explain this by observing that, to obtain the coefficient of x^2a, we are finding the number of ways of selecting two x's and one a from the three brackets $(x+a)(x+a)(x+a)$. So we are selecting two x's from the three brackets in every possible way, and taking an 'a' from the remaining bracket.

The number of ways of selecting two 'x's from three brackets is

$$^3C_2 = \frac{3!}{2!\ 1!} = 3.$$

Hence the coefficient of x^2a is 3.

Similarly, the number of ways of choosing one 'x' from $(x+a)(x+a)(x+a)$ is $^3C_1 = 3$. Hence the coefficient of xa^2 is 3.

In general, the number of ways of choosing r 'x's (and so automatically $n-r$ 'a's) from n brackets $(x+a)(x+a)\cdots(x+a)$ is ^nC_r.

The coefficient of $x^r a^{n-r}$ in the expansion of $(x+a)^n$ is therefore ^nC_r. For this reason the expression $^nC_r$is called the **binomial coefficient**.

Example

Find the coefficients of x^3a^2, x^4a, and x^5 in the expansion of $(x+a)^5$.

Solution

The coefficient of x^3a^2 is $^5C_3 = 10$.

The coefficient of x^4a is $^5C_4 = 5$.

The coefficient of x^5 is $^5C_1 = 1$.

We can check that these are correct by expanding $(x+a)^5$. Compare also with the fifth row of Pascal's triangle.

6.1 The binomial theorem

You may be familiar with the binomial theorem which is stated below. We will not prove this theorem but the preceding discussion may help you understand it.

For an integer n greater than 0,

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}a^3 + \cdots + nxa^{n-1} + a^n.$$

Notes

1. One way of proving the binomial theorem is by mathematical induction. If you are interested refer to the Mathematics Learning Centre's publication: Mathematical Induction.

2. If you are familiar with Σ notation, the theorem can be expressed more compactly as:

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^r a^{n-r}.$$

6.2 Exercises

1. Use the binomial theorem to expand $(x+a)^3$ and $(x+a)^4$.

2. **a.** Find the coefficients of x^6a and x^2a^5 in the expansion of $(x+a)^7$.

b. Expand Pascal's triangle up to $(x+a)^7$ and check the answers to a. against Pascal's triangle.

3. **a.** Simplify ${}^nC_0$, ${}^nC_1$, ${}^nC_2$ and ${}^nC_{n-2}$.

b. In the expression of $(x+a)^n$, which terms have coefficients ${}^nC_2$ and ${}^nC_{n-2}$?

4. In the expression of $(x-1)^8$, what is the coefficient of x^3?

5. Expand $(x-\frac{1}{2})^4$ fully.

7 Review problems

Try these problems. It will help you if you attempt to do them on your own rather than just following the solutions. Use them as a way of assessing how well you have understood the material in this book.

1. Eight birds are to be placed in eight different cages. How many arrangements are possible if each bird is placed in a separate cage.

2. Decide whether each of the following problems involves a permutation or a combination and then work out the answer.

a. How many 4 digit numbers can be made from the digits 2, 3, 5, 6, 7 and 9 if no repetition of digits is allowed.

b. A student has to answer 8 out of 10 questions in an exam. How many different choices has she?

c. How many different car number plates can be made if each plate contains 3 different letters followed by 3 distinct digits?

d. How many ways are there of playing a game of lotto requiring you to select 6 correct numbers out of 44?

3. **a.** How many committees of 4 people can be chosen from 5 men and 3 women?

b. How many of these could be all men?

c. How many would consist of 2 men and 2 women?

4. Find n if ${}^nP_2 = 56$.

5. **a.** How many arrangement of the letters in the word 'success' are there?

b. How many of these begin with an 's'?

6. Find the value of r if ${}^{11}C_r = 3 \times {}^{11}C_{r-1}$.

7. Find the coefficient of x^4 in the expansion of $(3x-\frac{1}{2x})^{20}$.

8 Solutions to exercises and review problems

Solution to exercises 2.1

1. There are 4 different colours for the paper and for each choice of paper there are two choices for the ribbon colour. Thus there are $4 \times 2 = 8$ colour combinations.

2. There are $2 \times 4 \times 6 \times 3$ different groups of trees, ie 144 groups. If the cherry has already been chosen, there are 48 ways of choosing the other varieties.

3. There are three possible results for the first match and, for each of these, three results for the second. Hence for two matches, there are 3^2 possible results. Continuing in this way we see that there are $3^4 = 81$ different sequences possible for the month.

Solution to exercises 3.1

1. **a.** $\frac{7!}{5!\,3!} = \frac{7.6.5!}{(5!)(3.2)} = \frac{7.6}{3.2} = 7$

b. $\frac{15!}{9!\,6!} = \frac{15.14.13.12.11.10.9!}{(9!)(6.5.4.3.2.1)} = 5005$ Cancel!

2. **a.** 362 880

b. $2.631308369 \times 10^{35}$

3. 69! is the largest factorial my calculator will evaluate. You may get a different answer depending on the type of calculator you have.

4.

5. **a.** $m! = m(m-1)(m-2)\cdots 2.1$

b. $\frac{m!}{r!} = \frac{m(m-1)(m-2)\cdots 2.1}{r(r-1)(r-2)\cdots 2.1}$

c. $(m-r)! = (m-r)(m-r-1)(m-r-2)\cdots 2.1$

d. $\frac{m(m-1)(m-2)\cdots 2.1}{[r(r-1)(r-2)\cdots 2.1][(m-r)(m-r-1)(m-r-2)\cdots 2.1]}$

Solutions to exercises 4.1

1. $\frac{20!}{15!}, \frac{7!}{5!}$

2. $\frac{6!}{2!} = \frac{6.5.4.3.2!}{2!} = 360,$

$\frac{6!}{4!} = \frac{6.5.4!}{4!} = 30.$

3. The number of ways of arranging six flags on the mast three at a time is ${}^6P_3 = \frac{6!}{3!} = 120.$ Each different order of the flags on the mast gives a different signal.

4. Any question along the lines of "How many ways are there of choosing 2 objects from 7 when order does matter". For example, How many ways are there of choosing a first and middle name for a baby out of 7 possible names? Or, How many ways are there of making up two letter 'words' from the letters a, b, c, d, e, f and g using each letter once only?

5.

6. **a.** ${}^mP_r = \dfrac{m!}{(m-r)!}$

b. ${}^nP_k = \dfrac{n!}{(n-k)!}$

c. ${}^nP_{n-k} = \dfrac{n!}{[n-(n-k)]!} = \dfrac{n!}{k!}$

Solutions to exercises 5.1

1. ${}^7C_3 = \dfrac{7!}{3!\;4!} = \dfrac{7.6.5.4!}{3.2.1.4!} = 35$

${}^7C_4 = \dfrac{7!}{4!\;3!} = 35$

${}^7C_0 = \dfrac{7!}{7!\;0!} = 1$ since 0! is defined to be 1.

2. $\dfrac{{}^9C_5}{{}^9C_6} = \dfrac{9!}{4!\;5!} \times \dfrac{3!\;6!}{9!} = \dfrac{3}{2}$

3. **a.** Numerator is $t(t-1(t-2)(t-3)$ as

$$\begin{aligned} {}^tC_4 &= \frac{t!}{(t-4)!4!} \\ &= \frac{t(t-1)(t-2)(t-3)(t-4)!}{(t-4)!4!} \\ &= \frac{t(t-1)(t-2)(t-3)}{4.3.2.1} \end{aligned}$$

b. Numerator is $7.6\cdots(7-t+1)$ as

$$\begin{aligned} {}^7C_t &= \frac{7!}{(7-t)!t!} \\ &= \frac{7.6\cdots(7-t+1)(7-t)!}{(7-t)!t!} \\ &= \frac{7.6\cdots(7-t+1)}{t(t-1)\cdots 1} \end{aligned}$$

4.

5. ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

${}^nC_{n-r} = \dfrac{n!}{[n-(n-r)]!(n-r)!} = \dfrac{n!}{r!(n-r)!}$

Notice that ${}^nC_r$ is the same as ${}^nC_{n-r}$.

6. The number of ways of choosing 5 pieces of fruit from 12 is

${}^{12}C_5 = \dfrac{12!}{5!\,7!} = 792.$

This is the same as ${}^{12}C_7$, the number of ways of choosing 7 pieces of fruit for the other basket.

7. Any question is acceptable where you are asked to choose 2 items from 7 items. For example, how many ways are there of choosing 2 chocolates from a box containing 7 different types of chocolate?

Solutions to exercises 6.2

1.

$$\begin{aligned}(x+a)^3 &= x^3+3x^2a+3xa^2+a^3\\ (x+a)^4 &= x^4+4x^3a+\frac{4(3)}{2!}x^2a^2+4xa^3+a^4\\ &= x^4+4x^3a+6x^2a^2+4xa^3+a^4\end{aligned}$$

2. **a.** The coefficient of x^6a in the expansion of $(x+a)^7$ is ${}^7C_6 = 7$. The coefficient of x^2a^5 is ${}^7C_2 = 21$.

b. The coefficents for the line $(x+a)^7$ in Pascal's triangle are 1, 7, 21, 35, 35, 21, 7, 1.

3. **a.** ${}^nC_0 = 1$, ${}^nC_1 = n$, ${}^nC_2 = \dfrac{n!}{(n-2)!2!} = \dfrac{n(n-1)}{2}$,

${}^nC_{n-2} = \dfrac{n!}{[n-(n-2)]!(n-2)!} = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)}{2}$.

b. The terms $x^{n-2}a^2$ and x^2a^{n-2} both have the coefficient ${}^nC_2 =^n C_{n-2}$.

4. The coefficient of x^3 in the expansion of $(x-1)^8$ is ${}^8C_3(-1)^5 = -56$.

5.

$$\begin{aligned}(x-\frac{1}{2})^4 &= x^4+4x^3(-\frac{1}{2})+6x^2(-\frac{1}{2})^2+4x(-\frac{1}{2})^3+(-\frac{1}{2})^4\\ &= x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}.\end{aligned}$$

Solutions to review problems

1. The first bird can be placed in any of the eight cages, the second in any of the 7 remaining cages and so on. By the counting principle there are $8\times7\times6\times5\times4\times3\times2\times1 = 8!$ ways of arranging the birds, that is 40320 ways.

2. **a.** ${}^6P_4 = \frac{6!}{2!} = \frac{6.5.4.3.2!}{2!} = 360$

b. ${}^{10}C_8 = \frac{10!}{2!\ 8!} = \frac{10.9.8!}{2.1\ 8!} = 45$

c. ${}^{26}P_3 \times {}^{10}P_3 = \frac{26!}{23!} \times \frac{10!}{7!} = \frac{26.25.24.23!}{23!} \times \frac{10.9.8.7!}{7!} = 11232000$

d. ${}^{44}C_6 = \frac{44!}{38!\ 6!} = \frac{44.43.42.41.39.38!}{38!\ 6.5.4.3.2.1} = 7059052.$

3. **a.** ${}^8C_4 = \frac{8!}{4!\ 4!} = \frac{8.7.6.5.4!}{4.3.2.1.4!} = 70$

b. ${}^5C_4 = \frac{5!}{1!\ 4!} = 5$

c. ${}^5C_2 \times {}^3C_2 = \frac{5!}{3!\ 2!} \times \frac{3!}{1!\ 2!} = 30.$

4. ${}^nP_2 = \frac{n!}{(n-2)!} = 56$, so $n! = 56(n-2)!$.

ie $n(n-1)(n-2)! = 56(n-2)!$, so $n^2 - n = 56$.

That is, $n^2 - n - 56 = 0$ therefore $(n-8)(n+7) = 0$, so $n = 8$ (as $n > 2$).

5. **a.** There are 7 letters in 'success' so at first glance there are 7! arrangements. However, interchanging the 3 's's will not affect the arrangement and neither will interchanging the two 'c's. Thus there are $\frac{7!}{3!\ 2!} = \frac{7.6.5.4.3!}{3!\ 2.1} = 420$ arrangements.

b. There are 6 letters left after placing the first 's'. Thus there are $\frac{6!}{2!\ 2!}$ ways of arranging the other letter, as we divide by the number of 'c's and the two remaining 's's. Therefore the number of ways $= \frac{6!}{2!\ 2!} = 180$.

6. ${}^{11}C_r = \frac{11!}{(11-r)!\ r!}$ and ${}^{11}C_{r-1} = \frac{11!}{(11-[r-1])!\ (r-1)!}$.

Now $\frac{11!}{(11-r)!\ r!} = \frac{3.11!}{(12-r)!\ (r-1)!}$,

ie $\frac{(12-r)!}{(11-r)!} = \frac{3.r!}{(r-1)!}$ dividing both sides by 11! and cross multiplying,

ie $\frac{(12-r)(11-r)!}{(11-r)!} = \frac{3r(r-1)!}{(r-1)!}$.

[Note: $(12-r)! = (12-r)(12-r-1)! = (12-r)(11-r)!$]

So, $12 - r = 3r$, ie $r = 3$.

You should check that ${}^{11}C_3 = 3 \times {}^{11}C_2$.

7. The term in x^4 is ${}^{20}C_{12}(3x)^{12}(\frac{-1}{2x})^8$.

So of the coefficient of x^8 is ${}^{20}C_{12}(3)^{12}(-\frac{1}{2})^8 = 261506338.9$. (Your calculator may round this off to 2.61506×10^8.)