Discrete-Time Complex Exponential Sequence.

\[ x[n] = C \alpha^n, \]

where C and \( \alpha \) are in general complex numbers. Alternatively we can express the sequence in the following form:

\[ x[n] = C e^{\beta n}, \]

where \( \alpha = e^\beta \).

Although this form is similar to the continuous-time exponential signal we have described previously, the former form is preferred when dealing with the discrete-time sequence.
Real Exponential Signal

\[ x[n] = C \alpha^n \] where \( \alpha > 1 \)  

\[ \text{e.g. } x[n] = 2 \times 1.1^n \]
Real Exponential Signal

\[ x[n] = C \alpha^n \text{ where } 0 < \alpha < 1 \quad x[n] = 2 \times 0.9^n \]
Real Exponential Signal

\[ x[n] = C \alpha^n \text{ where } -1 < \alpha < 0 \quad x[n] = 2 \times (-0.9)^n \]
Real Exponential Signal

\[ x[n] = C \alpha^n \quad \text{where} \quad \alpha < -1 \quad x[n] = 2(-1.1)^n \]
Real Exponential Signal

\[ x[n] = C \alpha^n \text{ where } \alpha = -1 \quad x[n] = 2 \alpha^n \]
Real Exponential Signal

Real-valued discrete exponentials are used to describe:-

1) Population growth as function of generation.

2) Total return on investment as a function of day, month or quarter.
Sinusoidal Signals

\[ x[n] = Ce^{\beta n}, \text{ let } C = 1 \& \beta = j\omega_o \text{ be purely an imaginary number.} \]

\[ \therefore x[n] = e^{j\omega_on}. \]

This signal is closely related to sinusoidal signal:
\[ x[n] = A\cos(\omega_o n + \phi). \]

Taking \( n \) as dimensionless, then both \( \omega_o \) and \( \phi \) have units of radians.

From Euler's relation: \( -e^{j\omega_on} = \cos \omega_on + j \sin \omega_on \)

\[ A\cos(\omega_on + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_on} + \frac{A}{2} e^{-j\phi} e^{-j\omega_on} \]
Discrete-time Sinusoidal Signals

\[ x[n] = \cos\left(\frac{2\pi n}{12}\right) \]

\[ x[n] = \cos\left(\frac{8\pi n}{31}\right) \]

\[ x[n] = \cos\left(\frac{n}{6}\right) \]
Discrete-time Sinusoidal Signals

These discrete-time signals possessed:-
1) Infinite total energy
2) Finite average power.
General Complex Exponential Signals

The general discrete-time complex exponential can be interpreted in terms of real exponentials and sinusoidal signals.

Writing C and $\alpha$ in polar form:

$C = |C| e^{j\theta}, \ \alpha = |\alpha| e^{j\omega_0}$.

$x[n] = C\alpha^n = |C| |\alpha|^n \cos(\omega_0 n + \theta) + j |C| |\alpha|^n \sin(\omega_0 n + \theta)$

$|\alpha| = 1$, real & imaginary parts are sinusoidal.

$|\alpha| < 1$, sinusoidal decaying exponentially,

$|\alpha| > 1$, sinusoidal growing exponentially.
General Complex Exponential Signals

$\alpha > 1$

$\alpha < 1$
Periodicity Properties of Discrete-time Complex Exponentials

Two properties of continuous-time counterpart $e^{j\omega_o t}$:

1) The larger is $\omega_o$, the higher is the rate of oscillation.
2) $e^{j\omega_o t}$ is periodic for any value of $\omega_o$.

There are differences in each of the above properties for the discrete-time case of $e^{j\omega_o n}$. 
Periodicity Properties of Discrete-time Complex Exponentials

Consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi}e^{j\omega_0n} = e^{j\omega_0n}.$$

From this we conclude that the exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency $\omega_0$. Similarly at frequencies $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$, and so on.

This is very different from the continuous-time case whereby the signals are all distinct for all distinct values of $\omega_0$. Because of this periodicity of $2\pi$, we need only to consider frequency interval of $2\pi$ in the case for discrete-time signals.
Periodicity Properties of Discrete-time Complex Exponentials

Because of this implied periodicity of discrete-time signal, the signal $e^{j\omega_{o}n}$ does not have a continually increasing rate of oscillation as $\omega_{o}$ is increased in magnitude. Increasing $\omega_{o}$ from 0 (d.c., constant sequence, no oscillation) the oscillation increases until $\omega_{o} = \pi$, thereafter the oscillation will decrease to 0 i.e. a constant sequence or d.c. signal at $\omega_{o} = 2\pi$. 
Periodicity Properties of Discrete-time Complex Exponentials

Therefore, low frequencies occur at \( \omega_0 = 0, \pm 2\pi, \pm \text{even multiple of } \pi. \)

High frequencies are at \( \omega_0 = \pm \pi, \pm 3\pi, \pm \text{odd multiple of } \pi. \)

Note for \( \omega_0 = \pi, \text{odd multiple of } \pi, \) \( e^{j\pi n} = (e^{j\pi})^n = (-1)^n, \)
the signal oscillates rapidly, changing sign at each point in time.
Periodicity Properties of Discrete-time Complex Exponentials

\[ x[n] = \cos(0n) = 1 \]
\[ x[n] = \cos(\pi n/8) \]
\[ x[n] = \cos(\pi n/4) \]
\[ x[n] = \cos(\pi n/2) \]
\[ x[n] = \cos(3\pi n/2) \]
\[ x[n] = \cos(7\pi n/4) \]
\[ x[n] = \cos(15\pi n/8) \]
\[ x[n] = \cos(2\pi n) \]
Periodicity Properties of Discrete-time Complex Exponentials

Second property concerns the periodicity of the discrete-time complex exponential.

In order for $e^{j\omega_0 n}$ to be periodic with period $N > 0$, $e^{j\omega_0(n+N)} = e^{j\omega_0 n}$, or equivalently $e^{j\omega_0 N} = 1$.

$\therefore \omega_0 N$ must be a multiple of $2\pi$.

i.e. $\omega_0 N = 2\pi m$, or equivalently $\frac{\omega_0}{2\pi} = \frac{m}{N}$,

This means that the signal $e^{j\omega_0 n}$ is periodic if $\omega_0 / 2\pi$ is a rational number and is not periodic otherwise.

This is also true for the discrete-time sinusoids.
Discrete-time Sinusoidal Signals

\[ x[n] = \cos(2\pi n / 12) \]
periodic because \( \omega_o = 2\pi / 12 \),
\[ \frac{\omega_o}{2\pi} = \frac{1}{12} \]

\[ x[n] = \cos(8\pi n / 31) \]
periodic because \( \omega_o = 8\pi / 31 \),
\[ \frac{\omega_o}{2\pi} = \frac{4}{31} \]

\[ x[n] = \cos(n / 6) \]
not periodic because \( \omega_o = 1 / 6 \),
\[ \frac{\omega_o}{2\pi} \neq \text{rational number} \]
Fundamental Period & Frequency of discrete-time complex exponential

If \( x[n] = e^{j\omega_0 n} \) is periodic with fundamental period \( N \),

Its fundamental frequency is \( \frac{2\pi}{N} = \frac{\omega_0}{m} \),

The fundamental period is written as :-

\[
N = m\left(\frac{2\pi}{\omega_0}\right)
\]
Comparison of the signal $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

$e^{j\omega_0 t}$
Distinct signals for distinct values of $\omega_0$.

Periodic for any choice of $\omega_0$.

Fundamental frequency $\omega_0$
Fundamental period
$\omega_0 = 0: undefined$

$\omega_0 \neq 0: \frac{2\pi}{\omega_0}$

$e^{j\omega_0 n}$
Identical signals for values of $\omega_0$ separated by multiples of $2\pi$

Periodic only if $\omega_0 = \frac{2\pi m}{N}$, for some integers $N > 0$ and $m$

Fundamental frequency $\frac{\omega_0}{m}$
Fundamental period
$\omega_0 = 0: undefined$

$\omega_0 \neq 0: m\left(\frac{2\pi}{\omega_0}\right)$
Harmonically related periodic exponential sequence

Considering periodic exponentials with common period N samples:
\[ \phi_k[n] = e^{jk(2\pi/N)n}, \text{ for } k = 0, \pm 1, \ldots \]

This set of signals possess frequencies which are multiples of
\[ 2\pi / N \]
Harmonically related periodic exponential sequence

In continuous-time case \( e^{jk(2\pi/T)t} \)

are all distinct signals for \( k=0, \pm 1, \pm 2, \ldots \)

\[
\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} \\
= e^{jk(2\pi/N)n} e^{j2\pi n} = \phi_k[n]
\]
Harmonically related periodic exponential sequence

Therefore, there are only $N$ distinct periodic exponentials in the discrete harmonic sequences.

$$\phi_o[n] = 1, \phi_1[n] = e^{j2\pi n/N}, \phi_2[n] = e^{j4\pi n/N},$$

$$\ldots \ldots \phi_{N-1}[n] = e^{j2\pi (N-1) n/N}$$

Any other $\phi_k[n]$ is identical to one of the above. (e.g. $\phi_N[n] = \phi_0[n]$)