Discrete-Time Fourier Transform

- Continuous-Time Fourier Transform
- Discrete-Time Fourier Transform
- Discrete-Time Fourier Transform Theorems.
- Band-Limited Discrete-Time Signals
- The Frequency Response of an LTI Discrete-Time System
- Phase and Group Delays
Continuous-Time Fourier Transform Pair Equation

Synthesis equation or Inverse FT.

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t} \, d\omega. \]

Analysis equation or Fourier Transform

\[ X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} \, dt. \]
Continuous-Time Fourier Transform

Inverse CTFT: \[ x(t) = \frac{1}{2\pi} \int_{\omega = -\infty}^{\omega = +\infty} X(j\omega) e^{j\omega t} d\omega \]

CTFT: \[ X(j\omega) = \int_{t = -\infty}^{t = +\infty} x(t) e^{-j\omega t} dt \]

\[ x(t) \leftrightarrow_{\text{CTFT}} X(j\omega) \]

\[ X(j\omega) = \text{Re}\{X(j\omega)\} + j \text{Im}\{X(j\omega)\} \]

\[ = |X(j\omega)| e^{j\angle X(j\omega)} \]

\(|X(j\omega)|\) is the magnitude spectrum.

\(\angle X(j\omega)\) is the phase spectrum

\[ = \arg\{X(j\omega)\} = \tan^{-1} \frac{\text{Im}\{X(j\omega)\}}{\text{Re}\{X(j\omega)\}} \]
Example 3.1

\[ x(t) = e^{-at} u(t) \]

\[ X(j\omega) = \int_{t=-\infty}^{t=+\infty} x(t)e^{-j\omega t} \, dt \]

\[ = \int_{0}^{\infty} e^{-at} e^{-j\omega t} \, dt = \int_{0}^{\infty} e^{-t(a+j\omega)} \]

\[ = \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \bigg|_{0}^{\infty} \]

\[ e^{-at} u(t) \xrightarrow{\text{CTFT}} \frac{1}{a+j\omega} \]
Convergence of CTFT

- CTFT of continuous-time signal will only exist if the Dirichlet conditions are satisfied:-
- (1) Signal must have finite number of discontinuities and finite number of maximum & minimum in any finite interval of time.
- (2) Signal must be absolutely integrable:-

\[ \int_{-\infty}^{\infty} |x(t)| dt < \infty \]
Properties of Fourier Transform

- Linearity
- Time Shifting
- Conjugation
- Differentiation in the time-domain
- Integration in the time-domain
- Time and Frequency Scaling.
- Duality
- Parseval’s Relation
- Convolution
- Multiplication
Summary of the approach to obtain the Fourier Transform for Discrete-time aperiodic signals.

\[ x[n] \text{ APERIODIC} \]

- construct periodic signal \( \tilde{x}[n] \) for which one period is \( x[n] \).

\( \tilde{x}[n] \) has a Fourier series.

- as period of \( \tilde{x}[n] \) increases,

\[ \tilde{x}[n] \rightarrow x[n], \]

and Fourier series of \( \tilde{x}[n] \rightarrow \) Fourier Transform of \( x[n] \)
Development of Discrete-time Fourier Transforms. DTFT.

Fourier Series of discrete-time periodic signal

\[ \tilde{x}[n] = \sum_{k=-N}^{N} a_k e^{jk(2\pi/N)n} \quad \text{Eqn 5.1} \]
Development of Discrete-time Fourier Transforms.

Thus we have the Discrete-time Fourier Transform pair as:

1) Synthesis equation, Inverse Fourier Transform:

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} \, d\omega. \]

2) Analysis equation, Fourier Transform or Fourier Spectrum:

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \]
Discrete-time Fourier Transform

\[ DTFT \]

\[ x[n] \leftrightarrow X(e^{j\omega}) \]

\[ X(e^{j\omega}) = \text{Re}\{X(e^{j\omega})\} + j\text{Im}\{X(e^{j\omega})\} \]

\[ = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})} \]

\[ |X(e^{j\omega})| \text{ is known as magnitude spectrum.} \]

\[ \angle X(e^{j\omega}) = \text{arg}\{X(e^{j\omega})\} = \tan^{-1} \frac{\text{Im}\{X(e^{j\omega})\}}{\text{Re}\{X(e^{j\omega})\}} \]

is known as the phase spectrum
Example 3.6 Discrete-time Fourier Transform.

\[ x[n] = a^n u[n], \quad 0 < a < 1 \]
Example 3.6 Discrete-time Fourier Transform.

\[ x[n] = a^n u[n], \quad 0 < a < 1 \]

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}} \]
Properties of Discrete-time Fourier Transforms.

\[
\text{DTFT} \quad x[n] \leftrightarrow X(e^{j\omega})
\]

\[
\text{DTFT} \quad x[-n] \leftrightarrow X(e^{-j\omega})
\]

**Periodic:**

\[
X(\omega) = X(\omega + 2\pi m)
\]
Properties of Discrete-time Fourier Transforms.

Conjugation:

\[ x^{*}[n] \leftrightarrow X^{*}(e^{-j\omega}) \]

Proof From: 

\[ -x[n] \leftrightarrow X(e^{j\omega}) \]

\[ x_{re}[n] + jx_{im}[n] \leftrightarrow X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}) \]

Taking conjugation on both side of above equation:

ie. replacing \( j = -j \).

\[ x_{re}[n] - jx_{im}[n] \leftrightarrow X_{re}(e^{-j\omega}) - jX_{im}(e^{-j\omega}) \]

\[ \therefore x^{*}[n] \leftrightarrow X^{*}(e^{-j\omega}). \]
Properties of Discrete-time Fourier Transforms.

Conjugation: $-x^*[n] \Leftrightarrow X^*(e^{-j\omega})$

Conjugate Symmetry: $-$

If $x[n]$ is real, then $x[n] = x^*[n] \Leftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

By Definition: $X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$

and $X^*(e^{-j\omega}) = X_{re}(e^{-j\omega}) - jX_{im}(e^{-j\omega})$

Since $X(e^{j\omega}) = X^*(e^{-j\omega})$ because $x[n]$ is real.

$\therefore X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}) = X_{re}(e^{-j\omega}) - jX_{im}(e^{-j\omega})$

$\therefore X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$, i.e. $X_{re}(e^{j\omega})$ is an even function of $\omega$.

$\therefore X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$, i.e. $X_{im}(e^{j\omega})$ is an odd function of $\omega$. 
Properties of Discrete-time Fourier Transforms.

Conjugate Symmetry

If $x[n]$ is real, then $x[n] = x^*[n] \leftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

By Defination:\[X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})\]

replacing $\omega \rightarrow -\omega$, \[X(e^{-j\omega}) = X_{re}(e^{-j\omega}) + jX_{im}(e^{-j\omega})\]

taking conjugation \[X^*(e^{-j\omega}) = X_{re}(e^{-j\omega}) - jX_{im}(e^{-j\omega})\]

\[|X(e^{j\omega})| = |X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})| = \sqrt{|X_{re}(e^{j\omega})|^2 + |X_{im}(e^{j\omega})|^2}\]

\[|X(e^{-j\omega})| = |X_{re}(e^{-j\omega}) + jX_{im}(e^{-j\omega})| = \sqrt{|X_{re}(e^{-j\omega})|^2 + |X_{im}(e^{-j\omega})|^2}\]

\[|X^*(e^{-j\omega})| = |X_{re}(e^{-j\omega}) - jX_{im}(e^{-j\omega})| = \sqrt{|X_{re}(e^{-j\omega})|^2 + |X_{im}(e^{-j\omega})|^2}\]

\[\therefore |X(e^{j\omega})| = |X(e^{-j\omega})| \quad \text{the function } |X(e^{j\omega})| \text{ is an even function of } \omega.

\[\arg\{X(e^{j\omega})\} = \tan^{-1}\frac{X_{im}(e^{j\omega})}{X_{re}(e^{j\omega})}\]

\[\arg\{X(e^{-j\omega})\} = \tan^{-1}\frac{X_{im}(e^{-j\omega})}{X_{re}(e^{-j\omega})}\]

\[\arg\{X^*(e^{-j\omega})\} = \tan^{-1}\frac{-X_{im}(e^{-j\omega})}{X_{re}(e^{-j\omega})} = -\tan^{-1}\frac{X_{im}(e^{-j\omega})}{X_{re}(e^{-j\omega})}\]

\[\therefore \arg\{X^*(e^{-j\omega})\} = -\arg\{X(e^{-j\omega})\}\]

and Since $X(e^{j\omega}) = X^*(e^{-j\omega})$ because $x[n]$ is real.

\[\arg\{X(e^{j\omega})\} = \arg\{X^*(e^{j\omega})\}\]

\[\therefore \arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}\]

i.e Phase angle $X(e^{j\omega})$ is an odd function of $\omega$. 
Properties of Discrete-time Fourier Transforms.

If $x[n]$ is real, and $x[n] \leftrightarrow X(e^{j\omega}) = \text{Re}\{X(e^{j\omega})\} + j\text{Im}\{X(e^{j\omega})\}$

$x[n] = \text{Even part } x[n] + \text{Odd part of } x[n]$

$X(e^{j\omega}) = \text{Even part } X(e^{j\omega}) + \text{Odd part } X(e^{j\omega})$.

But from previous slide, \text{Real} \{X(e^{j\omega})\} is even function of $\omega$.

\[ \therefore \text{Ev}\{x[n]\} \leftrightarrow \text{Re}\{X(e^{j\omega})\} \]

From previous slide, \text{Im} \{X(e^{j\omega})\} is odd function of $\omega$.

\[ \therefore \text{Odd}\{x[n]\} \leftrightarrow j\text{Im}\{X(e^{j\omega})\} \]
Discrete-time Fourier Transform Theorems.

Time shifting:

\[ x[n - n_0] \leftrightarrow e^{-j\omega_0} X(e^{j\omega}) \]

Frequency shifting:

\[ e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)}) \]

Linearity:

\[ ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega}) \]

Parseval's relation:

\[ \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega \]
Convolution Theorem

\[ h[n] * x[n] \leftrightarrow H(\omega)X(\omega) \]
Multiplication Theorem

\[ h[n]x[n] \leftrightarrow \frac{1}{2\pi} H(\omega) * X(\omega) \]
Band-Limited Discrete-Time Signals. (e.g. Bandpass)

\[ X(j\omega) = 0, \quad \text{for } 0 \leq |\omega| < \omega_a \text{ and } \omega_b \leq |\omega| < \frac{\pi}{2} \]

Bandwidth = \( \omega_b - \omega_a \)
Lowpass Discrete-Time Signals

\[ X(j\omega) = 0, \quad \omega_p \leq |\omega| < \pi \]

\[ \neq 0, \quad 0 \leq |\omega| < \omega_p \]

Bandwidth = \( \omega_p \)
Highpass Discrete-Time Signals

\[ X(j\omega) \neq 0, \quad \omega_p \leq |\omega| < \pi \]

\[ = 0, \quad 0 \leq |\omega| < \omega_p \]

Bandwidth = \( \pi - \omega_p \).
e.g. Highpass Discrete-Time real signal
Frequency Response of LTI DTS

\[ x[n] = e^{j\omega n} \]

\[ y[n] = H(e^{j\omega})e^{j\omega n} \]

\[ y[n] = x[n] * h[n] \]

In the time domain the output \( y[n] \) is calculated, by convolving input sequence with the system impulse response:

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \]

Important property of LTI system, certain input signals known as the eigenfunction produces the same function at the output of the system multiplied with complex constant known as eigenvalue.
Frequency Response of LTI DTS

\[ x[n] = e^{j\omega n} \]

\[ y[n] = H(e^{j\omega})e^{j\omega n} \]

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \]

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \]

\[ y[n] = H(e^{j\omega})e^{j\omega n} \]

where \( H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \)

This is the DTFT of \( h[n] \) by changing the variable \( k \) to \( n \).

\( H(e^{j\omega}) \) is the frequency response of the LTI DTS and is related to its impulse response \( h[n] \) through DTFT.

\[ \text{DTFT} \quad i.e. \ h[n] \Leftrightarrow H(e^{j\omega}) \]
Frequency Response of LTI DTS

Time Domain

\[ x[n] \quad \Downarrow \quad X(e^{j\omega}) \]

\[ h[n] \quad \Downarrow \quad H(e^{j\omega}) \]

\[ y[n] \quad \Downarrow \quad Y(e^{j\omega}) \]

Frequency Domain

Time Domain \( y[n] = x[n] * h[n] \)

But by DTFT theorem:

- convolution in time domain \( \equiv \) multiplication in frequency domain

\[ x[n] * h[n] \overset{DTFT}{\iff} X(e^{j\omega})H(e^{j\omega}). \]

\[ y[n] \overset{DTFT}{\iff} Y(e^{j\omega}). \quad \text{i.e.} \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \]
Frequency Response LTI DTS

\( H(e^{j\omega}) \) is a complex function and is known as Frequency Response.

\[
H(e^{j\omega}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega})
\]

\[
= |H(e^{j\omega})| e^{j\theta(\omega)}
\]

\[
|H(e^{j\omega})| = \sqrt{H_{re}^2(e^{j\omega}) + H_{im}^2(e^{j\omega})} \quad \text{Magnitude Response}
\]

\[
\theta(\omega) = \arg\{H(e^{j\omega})\} \quad \text{Phase Response}
\]

\[
G(\omega) = 20\log_{10}|H(e^{j\omega})| \quad \text{decibels (dB)} \quad \text{Gain Function}
\]

\[
A(\omega) = -G(\omega) \quad \text{Attenuation or Loss Function}
\]
Frequency Response of LTI FIR Discrete-Time System

Input - Output Relationship given by:

\[ y[n] = \sum_{k=N_1}^{N_2} h[k]x[n - k], \quad N_1 < N_2. \]

Taking DTFT with linearity & time-shifting properties:

\[ Y(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}X(e^{j\omega}), \]

\[ \therefore \text{Frequency Response} \ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k} \]

\[ H(e^{j\omega}) \text{ is a polynomial in } e^{j\omega}. \]
Frequency Response of LTI IIR Discrete-Time System

Input - Output Relationship given by:

\[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]. \]

Taking DTFT with linearity & time-shifting properties:

\[ \sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} X(e^{j\omega}). \]

\[ \therefore \text{Frequency Response} \ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \]

\( H(e^{j\omega}) \) is a rational function in \( e^{j\omega} \).
Concept of Filtering

- Discrete Signals are sequences of numbers in time-domain.
- Through DTFT, Discrete Signals are transformed into frequency domain which are represented as magnitude and phase spectrum of frequencies.
- Shape the Spectrums to the desired forms by multiplying them with frequency response of LTI Discrete-Time Systems.
  - Frequency shaping filters
  - Frequency selective filters.
Frequency Response of LTI DTS

Time Domain

\[ x[n] \quad \overset{\text{DTFT}}{\leftrightarrow} \quad X(e^{j\omega}) \]

\[ h[n] \quad \overset{\text{DTFT}}{\leftrightarrow} \quad H(e^{j\omega}) \]

\[ y[n] \quad \overset{\text{DTFT}}{\leftrightarrow} \quad Y(e^{j\omega}) \]

Frequency Domain

Time Domain \( y[n] = x[n] \ast h[n] \)

But by DTFT theorem:

Convolution in time domain \( \equiv \) multiplication in frequency domain

\[ x[n] \ast h[n] \Leftrightarrow X(e^{j\omega})H(e^{j\omega}). \]

\[ y[n] \Leftrightarrow Y(e^{j\omega}). \quad \text{i.e.} \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}. \]
In Frequency Domain

Signal Magnitude Spectrum

Magnitude Response Of LTI DTS (Filter)

\[ X(j\omega) = 0, \quad \omega_p \leq |\omega| < \pi \]
\[ \neq 0, \quad 0 \leq |\omega| < \omega_p \]

Bandwidth = \( \omega_p \)
What happen to Phase?

- Phase and Group Delays need to be considered since both DTFT of Input and impulse response are complex functions.
- Multiplying these two complex functions will result in addition of their phase components (argument of complex function)
- Group delay is defined as the negative derivative of phase function with respect to angular frequency.
Phase & Group Delay

\[ Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \]

\[ \angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega}) \]

IF \( \angle H(e^{j\omega}) \) is denoted as \( \theta(\omega) \)

IF \( \theta(\omega) < 0 \), the output will lag the input.

IF \( \theta(\omega) > 0 \), the output will lead the input.

Group Delay = \[-\frac{d\theta(\omega)}{d\omega}\]

Phase Delay = \[-\frac{\theta(\omega_0)}{\omega_0}\]