z- Transforms

- Frequency domain representations of discrete-time signals and LTI discrete-time systems are made possible with the use of DTFT.
- However not all discrete-time signals (e.g. unit step sequence) are guaranteed to have DTFT because of convergence condition.
- As a result for these cases we are not able to use such frequency domain characterization.
Definition of z-Transform.

DTFT is given by -

\[ G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}. \]

Certain function of \( g[n] \) will not allow the above summation to converge. Therefore we need to find a way of overcoming this non-convergence. Let us modified the signal \( g[n] \) to \( g[n]r^{-n} \)

DTFT of \( g[n]r^{-n} \) is \( G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n} = \sum_{n=-\infty}^{\infty} g[n](re^{j\omega})^{-n}. \)

Letting \( re^{j\omega} = z \), therefore \( G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} \ldots Eqn.6.1 \)
Region of Convergence.

Another way looking at the transform $G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$. 

$z$-transform is the DTFT of modified signal $g[n]r^{-n}$. 

where $z = re^{j\omega}$. 

By choosing the right value of $r$, 
we can make the summation above to converge. 

The values of $r$ that make this convergence possible 
is known as Region of Convergence (ROC). 

ROC are circles or annular rings in the $z$-plane.
Point $z$ in a Complex $z$-plane

$z = re^{j\omega}$

Real Axis

Imaginary Axis

Unit Circle
Annular ring in a Complex z-plane

$z = re^{j\omega}$

Unit Circle

Real Axis

Imaginary Axis

ROC

$1$

Unit Circle
Example 6.1 z-Transform of Causal Exponential Sequence.

What is the z-transform of \( x[n] = \alpha^n \mu[n] \)?

Using eqn. 6.1, \( G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} \)

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n.
\]

(The above summation is a sum of geometric series)

\[
.: X(z) = \frac{1}{1-(\alpha z^{-1})}, \text{ only if } |\alpha z^{-1}| < 1.
\]

The z-transform here is an algebraic expression but must be accompany with the condition of convergence or else it is not valid.

ROC here is outside a circle(annular region) \(|z| > |\alpha|\).
\[ r^{-n} = 1.2^{-n} \]

\[ x[n] = \alpha^n \mu[n] = (1.1)^n \mu[n]. \]
Real Axis

Imaginary Axis

ROC

Unit Circle

$|\alpha| < 1$
Unit step sequence

If $\alpha = 1$, $x[n] = 1^n \times \mu[n] = \mu[n]$.

DTFT of $\mu[n] = \sum_{n=-\infty}^{\infty} \mu[n]e^{-j\omega n}$

$= \sum_{n=0}^{\infty} 1 \times e^{-j\omega n} \Rightarrow \infty$(DTFT does not exit)

The unit step sequence is not absolutely summable, hence it DTFT does not converge.

But $z$-transform of $\mu[n]$ :-

$X(z) = \frac{1}{1 - (\alpha z^{-1})} = \frac{1}{1 - z^{-1}}$,

only if $|z^{-1}| < 1$ i.e. $|z| > 1.$ (ROC).
Since unit circle is not in ROC, DTFT of unit step does not converge.
Example 6.2 z-Transform of Anti-causal Exponential Sequence.

What is the z - transform of \( x[n] = -\alpha^n \mu[-n - 1] \)? {left hand sequence}

Using eqn. 6.1, \( X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} -\alpha^n \mu[-n - 1]z^{-n} = -\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n \).

\[
= -\sum_{n=-1}^{-\infty} (\alpha z^{-1})^n = -\sum_{m=1}^{\infty} (\alpha z^{-1})^{-m} = -\sum_{m=0}^{\infty} (\alpha z^{-1})^{-m} + 1 = 1 - \sum_{m=0}^{\infty} (\alpha^{-1} z)^m
\]

\[
= 1 - \frac{1}{1 - (\alpha^{-1} z)}, \quad \text{provided that } |(\alpha^{-1} z)| < 1.
\]

\[
X(z) = \frac{-(\alpha^{-1} z)}{1 - (\alpha^{-1} z)} = \frac{1}{1 - \frac{1}{1 - (\alpha^{-1} z)}} = \frac{1}{1 - (\alpha z^{-1})}, \quad \text{provided that } |(\alpha^{-1} z)| < 1.
\]

The end algebraic expression is exactly the same as in example 6.1, the only that is different here is ROC |(\alpha^{-1} z)| < 1.

\( \therefore \text{ROC is inside a circle(annular region)} | z | < | \alpha | \)
The diagram illustrates the region of convergence (ROC) for a complex function. The ROC is defined by the condition $|\alpha| < 1$, where $\alpha$ is a complex number. The shaded area inside the circle of radius 1 but outside the unit circle represents the ROC. The unit circle is denoted by the label "Unit Circle."
**z-Transform of Finite Sequences**

\[ G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} \]

If \( g[n] \) is finite e.g. \( g[n] = [1,-2,3,-1,0,3,7] \).

\[ G(z) = \sum_{n=-3}^{3} g[n]z^{-n} = 1z^{3} + (-2)z^{2} + 3z^{1} + (-1)z^{0} + 0z^{-1} + 3z^{-2} + 7z^{-3} \].

ROC for this \( G(z) \) is the entire \( z \) - plane except for \( z = 0 \) and \( z = \infty \)

If \( g[n] \) is finite e.g. \( g[n] = [-1,0,3,7] \).

\[ G(z) = \sum_{n=0}^{3} g[n]z^{-n} = (-1)z^{0} + 0z^{-1} + 3z^{-2} + 7z^{-3} \].

ROC for this \( G(z) \) is the entire \( z \) - plane except for \( z = 0 \).

If \( g[n] \) is finite e.g. \( g[n] = [1,-2,3,-1] \)

\[ G(z) = \sum_{n=-3}^{0} g[n]z^{-n} = 1z^{3} + (-2)z^{2} + 3z^{1} - 1. \]

ROC for this \( G(z) \) is the entire \( z \) - plane except for \( z = \infty \).
Rational z-transforms

Ratio of two polynomials $P(z)$ and $D(z)$ in $z^{-1}$ :

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}} \quad \text{Eqn.6.13}$$

Alternate Representation as ratio of polynomials in $z$ :

$$H(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + p_2 z^{M-2} \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + d_2 z^{N-2} \cdots + d_{N-1} z + d_N} \quad \text{Eqn.6.14}$$

Factoring above equations :

$$H(z) = \frac{p_0}{d_0} \frac{\prod_{l=1}^{M} (1 - \xi_l z^{-1})}{\prod_{l=1}^{N} (1 - \lambda_l z^{-1})} = z^{(N-M)} \frac{p_0}{d_0} \frac{\prod_{l=1}^{M} (z - \xi_l)}{\prod_{l=1}^{N} (z - \lambda_l)} \quad \text{Eqn.6.15}$$

$z = \xi_l$ are $M$ zeros of $H(z)$ on $z$ - plane. i.e. obtained by taking $H(z) = 0$.

$z = \lambda_l$ are $N$ poles of $H(z)$ on $z$ - plane. i.e. obtained by taking $H(z) = \infty$.

If $N > M$, there are additional $(N - M)$ zeros at $z = 0$ (origin of $z$ - plane).

If $N < M$, there are additional $(M - N)$ poles at $z = 0$ (origin of $z$ - plane).
Examples of Rational z-Transforms

(1)  \( z \)-transform of \( \mu[n] \): –

\[
\mu(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \quad \text{for } |z| > 1. \text{(ROC)}.
\]

having a zero at \( z = 0 \), and a pole at \( z = 1 \).

(2)  \( H(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}} \)

Conjugate poles at \( z = 0.4 \pm j0.6928 \),

Conjugate zeros at \( z = 1.2 \pm j1.2 \)
Discrete-Time Fourier Transform

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{-j\omega n} \, d\omega \]

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]

**LTI Systems Impulse response**

\[ e^{j\omega n} \rightarrow H(\omega) e^{j\omega n} \]

\[ X(\omega) = \mathcal{F}\{x[n]\} \]

\[ x[n] \leftrightarrow X(\omega) \]

\[ z^n \rightarrow \sum_{k=-\infty}^{+\infty} h[k] z^{-k} \]

\[ z = r e^{j\omega} \]

\[ H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} \]

\[ X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \]

\[ \mathcal{F}\{x[n]\} = X(z) \]

\[ \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{+\infty} \]
\[ X(z) \bigg|_{z = e^{j\omega}} = X(e^{j\omega}) \]

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] (re^{j\omega})^{-n} \]

\[ = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} \]

Example 10.1
\[ x[n] = a^n u[n] \]
\[ X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad |a| < 1 \]

\[ X(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} u[n] \]
\[ = \frac{1}{1 - a_3 z^{-1}} \quad |a_3^*| < 1 \]
\[ |a_r^*| < 1 \]

\[ a^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - a_3 z^{-1}} \quad |z| > |a| \quad \text{ROC} \]

Example 10.2
\[ -a^n u[n-1] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - a_3 z^{-1}} \quad |z| < |a| \quad \text{ROC} \]
The z-transform and associated pole-zero plot for a right-sided exponential sequence.

\[ a^n u[n] \quad Z \quad \frac{1}{1 - az^{-1}} \quad |z| > |a| \]
The $z$-transform and associated pole-zero plot for a left-sided exponential sequence.

$$-a^n u[-n-1] \leftrightarrow \frac{z}{1 - az^{-1}} \quad |z| < |a|$$

Diagram showing the unit circle and the $z$-plane.
Pole-zero plot for a discrete-time underdamped second-order system illustrating the geometric determination of the Fourier transform from the pole-zero plot.
Properties of the ROC of the $z$-transform.

PROPERTIES OF THE REGION OF CONVERGENCE

- The ROC does not contain poles
- The ROC of $X(z)$ consists of a ring in the $z$-plane centered about the origin
- $\mathcal{F}\{x[n]\}$ converges $\iff$ ROC includes the unit circle in the $z$-plane
- $x[n]$ finite duration $\Rightarrow$ ROC is entire $z$-plane with the possible exception of $z = 0$ or $z = \infty$
Properties of the ROC for a right-sided sequence.

- \( x[n] \) right-sided and \( |z| = r_o \) is in ROC
  \[ \Rightarrow \text{all finite values of } z \text{ for which } |z| > r_o \]
  are in ROC

- \( x[n] \) right-sided and \( X(z) \) rational
  \[ \Rightarrow \text{ROC is outside the outermost pole} \]
Properties of the ROC of the $z$-transform for a left-sided sequence and for a two-sided sequence.

- $x[n]$ left-sided and $|z| = r_o$ is in ROC => all values of $z$ for which $0 < |z| < r_o$ will also be in ROC

- $x[n]$ left-sided and $X(z)$ rational => ROC inside the innermost pole

- $x[n]$ two-sided and $|z| = r_o$ is in ROC => ROC is a ring in the $z$-plane which includes the circle $|z| = r_o$
ROC for right-sided sequence

\[ X(z) = \frac{z}{(z - \frac{1}{3})(z - 2)} \]
The ROC if the sequence is left-sided.

\[ X(z) = \frac{z}{(z - \frac{1}{3})(z - 2)} \]
The ROC if the sequence is two-sided.

\[ X(z) = \frac{z}{(z - \frac{1}{3})(z - 2)} \]
Inverse Transform

\[ \mathbf{X}(\mathbf{3}) = \mathcal{F}\{ x[\mathbf{n}] r^{-n} \} \]

\[ x[\mathbf{n}] r^{-n} = \mathcal{F}^{-1}\{ \mathbf{X}(\mathbf{3}) \} \]

\[ = \frac{1}{2\pi} \int \mathbf{X}(re^{j\mathbf{n}}) e^{j\mathbf{n} \cdot \mathbf{3}} d\mathbf{n} \]

\[ x[\mathbf{n}] = \frac{1}{2\pi} \int \int \mathbf{X}(re^{j\mathbf{3}})(re^{j\mathbf{3}})^n d\mathbf{3} \]

\[ 3 = re^{j\mathbf{n}} \quad d\mathbf{3} = ire^{j\mathbf{n}} d\mathbf{n} \]

\[ x[\mathbf{n}] = \frac{1}{2\pi i} \oint \mathbf{X}(\mathbf{3}) 3^{n-1} d\mathbf{3} \]
Methods of computing the Inverse z-transform

- Using Cauchy’s Residue Theorem
- Table look-up
- Partial Fraction Expansion
- Power Series Expansion or Long Division.
Inverse z-Transform via Table Look-Up

Example 6.12.

\[ H(z) = \frac{0.5z}{z^2 - z + 0.25}, \quad |z| > 5. \]

\[ H(z) = \frac{0.5z}{(z - 0.5)^2} = \frac{0.5z^{-1}}{(1 - 0.5z^{-1})^2}. \]

From Table 6.1:

\[ n\alpha^n \mu[n] \Leftrightarrow \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \quad |z| > |\alpha|. \]

\[ \therefore h[n] = n(0.5)^n \mu[n] \]
Inverse z-Transform via Partial Fraction Expansion

Example 6.13.

\[ H(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}. \]

This is an improper function since \( M = 3 > N = 2 \).

Perform long division first by reversing order of both polynomials to get at the proper function.

\[
\begin{align*}
H(z) &= 0.2z^{-2} + 0.8z^{-1} + 1 \left( \frac{(0.3/0.2)z^{-1} - (0.7/0.2)}{0.3z^{-3} + 1.2z^{-2} + 1.5z^{-1}} \right) \\
&\quad \quad \quad - 0.7z^{-2} - 0.7z^{-1} + 2 \\
&\quad \quad \quad - 0.7z^{-2} - 2.8z^{-1} - 3.5 \\
&\quad \quad \quad \quad - 2.1z^{-1} + 5.5
\end{align*}
\]

\[
\therefore \ H(z) = 1.5z^{-1} - 3.5 + \frac{-2.1z^{-1} + 5.5}{0.2z^{-2} + 0.8z^{-1} + 1}.
\]
Continue Example 6.13

Now from the proper function \[
\frac{-2.1z^{-1} + 5.5}{0.2(z^{-2} + 4z^{-1} + 5)} = \frac{-10.5z^{-1} + 27.5}{(z^{-2} + 4z^{-1} + 5)}
\] and

the quadratic formula \[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\]

Roots of denominator of proper function = \[
\frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm j
\]

Performing the partial fraction expansion for the proper function:

\[
\frac{-2.1z^{-1} + 5.5}{0.2z^{-2} + 0.8z^{-1} + 1} = \frac{-10.5z^{-1} + 27.5}{(z^{-1} - (-2 + j))(z^{-1} - (-2 - j))}
\]
Continue Example 6.13

\[
= \frac{-10.5(-2+j)^{-1} + 27.5}{(z^{-1} - (-2+j))((-2+j)^{-1} - (2-j))} + \frac{-10.5(-2-j)^{-1} + 27.5}{((-2-j)^{-1} - (2-j))((2-j)^{-1} - (-2-j))}
\]

\[
= \frac{-10.5 + 27.5(-2+j)}{(z^{-1} - (-2+j))((1 - (-2-j)(-2+j))} + \frac{-10.5 + 27.5(-2-j)}{(1-(-2+j)(2-j))((2-j)^{-1} - (-2-j))}
\]

\[
= \frac{-65.5 + 27.5j}{(z^{-1} - (-2+j))(4)} + \frac{-65.5 - 27.5j}{(4)(z^{-1} - (-2-j))}
\]

\[
= \frac{16.375 - 6.875j}{(z^{-1} - (-2+j))} + \frac{16.375 + 6.875j}{(z^{-1} - (-2-j))}
\]

\[
\alpha^n \mu[n] \Leftrightarrow \frac{1}{(1-\alpha z^{-1})}, \quad |z| > |\alpha|. \quad \alpha = (-2+j)^{-1} and (-2-j)^{-1}
\]

\[
\therefore h[n] = -3.5\delta[n] + 1.5\delta[n-1] + \frac{16.375 - 6.875j}{(-2+j)}(-2+j)^n \mu[n] + \frac{16.375 + 6.875j}{(-2-j)}(-2-j)^n \mu[n]
\]
Partial Fractions

\[ X(3) = \frac{3}{(3^{-1})(3^{-2})} \quad 13 > 2 \]

\[ = \frac{3^{-1}}{(1-\frac{1}{3^2})(1-\frac{1}{3^3})} \quad 13 > 2 \]

\[ = \frac{-3/5}{(1-\frac{1}{3^2})} + \frac{3/5}{(1-\frac{1}{3^3})} \quad 13 > 2 \]

\[ = -\frac{3}{5} \left(\frac{1}{3}\right)^n u[n] + \frac{3}{5} \left(\frac{2}{3}\right)^n u[n] \]

Power Series

\[ X(3) = \sum_{n=-\infty}^{\infty} x[n] 3^n \]

\[ \frac{1}{1-a_3^{-1}} = 1 a_3^{-1} \frac{1}{1-a_3^{-1}} \]

\[ \frac{1}{1-a_3^{-1}} = 1 + a_3^{-1} + a_3^{-2} + \ldots \]

\[ x[n] = a^n u[n] \]
Inverse z-Transform Via Power Series Expansion/Long Division

Example 6.19

\[ H(z) = \frac{1 + 2.0z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}. \]
\[
\begin{align*}
&\quad 1 + 0.4z^{-1} - 0.12z^{-2} \\
&\quad 1 + 2.0z^{-1} \\
&\quad 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \ldots \ldots
\end{align*}
\]

\[
\begin{align*}
&\quad 1 + 2.0z^{-1} \\
&\quad 1 + 0.4z^{-1} - 0.12z^{-2} \\
&\quad 1.6z^{-1} + 0.12z^{-2} \\
&\quad 1.6z^{-1} + 0.64z^{-2} - 0.192z^{-3} \\
&\quad -0.52z^{-2} + 0.192z^{-3} \\
&\quad -0.52z^{-2} - 0.208z^{-3} + 0.0624z^{-4} \\
&\quad 0.400z^{-3} - 0.0624z^{-4} \\
&\quad 0.400z^{-3} + 0.1600z^{-4} - 0.0480z^{-5} \\
&\quad -0.2224z^{-4} + 0.0480z^{-5}
\end{align*}
\]

\[ H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \ldots \ldots \]

\[ \therefore h[n] = 1\delta[n] + 1.6\delta[n-1] - 0.52\delta[n-2] + 0.4\delta[n-3] - 0.2224\delta[n-4] + \ldots \ldots \]
Properties of z-Transform

- Linearity
- Time Shifting
- Scaling in the z-domain
- Time Reversal
- Time Expansion
- Conjugation
- Convolution
- Differentiation in the z-domain
- The initial-value Theorem.
z-T Property associated with Linearity

\[ x_1[n] \xleftarrow{z^{-T}} X_1(z) \text{ with ROC denoted by } R_1 \]

\[ x_2[n] \xleftarrow{z^{-T}} X_2(z) \text{ with ROC denoted by } R_2 \]

\[ ax_1[n] + bx_2[n] \xleftarrow{LT} aX_1(z) + bX_2(z) \text{ with ROC containing } R_1 \bigcap R_2. \]

\( \bigcap \) is the symbol for intersect with.
z-T Properties continued

\[ x[n] \xleftarrow{zT} X(z), \quad \text{with } ROC = R, \]

Time Shifting :-
then \[ x[n - n_0] \xleftarrow{zT} z^{-n_0} X(z), \quad \text{with } ROC = R, \text{except for possible addition or deletion of the origin or infinity.} \]

Scaling in the z - domain :-
then \[ z_0^n x[n] \xleftarrow{zT} X\left(\frac{z}{z_0}\right), \quad \text{with } ROC = |z_0| R. \]

Time Reversal :-
then \[ x[-n] \xleftarrow{zT} X\left(\frac{1}{z}\right), \quad \text{with } ROC = \frac{1}{R}. \]

Time Expansion :-
\[ x_{(k)}[n] = x[n/k] \text{ if } n \text{ is a multiple of } k, \]
\[ = 0 \text{ if } n \text{ is not a multiple of } k. \]
then \[ x_{(k)}[n] \xleftarrow{zT} X(z^k), \quad \text{with } ROC = R^{1/k}. \]

Conjugation :-
then \[ x^*[n] \xleftarrow{zT} X^*(z^*), \quad \text{with } ROC = R. \]
z-T Property associated with Convolution

\[ x_1[n] \xleftarrow{z^T} X_1(z) \text{ with ROC denoted by } R_1 \]

\[ x_2[n] \xleftarrow{z^T} X_2(z) \text{ with ROC denoted by } R_2 \]

\[ x_1[n] * x_2[n] \xleftarrow{z^T} X_1(z) X_2(z) \text{ with ROC containing } R_1 \bigcap R_2. \]

\( \bigcap \) is the symbol for intersect with.
zT Properties continued

\[ x[n] \xleftarrow{zT} X(z), \quad \text{with } \text{ROC} = \mathbb{R}, \]

Differentiation in the z - Domain : -

then \[ nx[n] \xleftarrow{zT} -z \frac{dX(z)}{dz}, \quad \text{with } \text{ROC} = \mathbb{R}. \]

Initial - value theorem : -
If \( x[n] = 0 \) for \( n < 0 \),
\[ \text{then } x[0] = \lim_{z \to \infty} X(z). \]
Analysis & Characterization of LTI systems using z-Transforms.

If \( x[n] = z^n \), \( y[n] = H(z).z^n \),

where \( z^n \) is eigenfunction of LTI system,

\( H(z) \) is eigenvalue of LTI system

\[ \begin{align*}
  X(z) &\rightarrow h[n] & h[n] &\rightarrow y[n]=x[n]*h[n] \\
  X(z) &\rightarrow H(z) & H(z) &\rightarrow Y(z)=X(z).H(z)
\end{align*} \]

\( H(z) \) is known as System Function or Transfer Function

Frequency response = \( H(z) \) with \( z = e^{j\omega} \) (unit circle in ROC)
Causality.

(1) A LTI system is causal if :-
Its impulse response \( h[n] = 0 \) for \( n < 0 \) ie. right - sided.
or in other words :-
ROC of its system function \( H(z) \) is the exterior of the circle including infinity.

(2) A discrete - time LTI system with rational system function \( H(z) \)
is causal if and only if :-
(a) the ROC is the exterior of a circle outside the outermost pole;
and (b) with \( H(z) \) expressed as a ratio of polynomials in \( z \),
the order of the numerator cannot be greater than the order of the denominator.
Stability.

- An LTI system is stable if and only if:-
  - Its Impulse response $h[n]$ is absolutely summable.
  - Or Fourier Transform of $h[n]$ converges.
  - Or the ROC of its system function $H(z)$ includes the unit circle, $|z|=1$

- For causal LTI system, all poles of $H(z)$ must be in the unit circle, $|z|=1$
Example 6.14 Inverse z-transform, Causality & Stability.

\[ H(z) = \frac{z(z + 2.0)}{(z - 0.2)(z + 0.6)} \]

Determine all the possible impulse responses \( h[n] \) for the given \( H(z) \) above.

Associate each one of the above impulse responses with the ROC, stability, and causality.

Using Partial-Fraction Expansion:

\[ H(z) = \frac{z(z + 2.0)}{(z - 0.2)(z + 0.6)} = \frac{2.75}{1 - 0.2z^{-1}} - \frac{1.75}{1 + 0.6z^{-1}}. \]

Poles are \( z = 0.2 \) and \( z = -0.6 \)

ROC 1: - Right hand impulse response \( h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n] \). Causal & Stable.

ROC 2: - R.H & L.H impulse response \( h[n] = 2.75(0.2)^n \mu[n] - 1.75(-1)(-0.6)^n \mu[-n-1] \). Not Causal & Not Stable.

ROC 3: - Left hand impulse response \( h[n] = 2.75(-1)(0.2)^n \mu[-n-1] - 1.75(1)(-0.6)^n \mu[-n-1] \). Not Causal & Not Stable.

\[
\text{ROC 1:} \quad |z| > 0.6 \\
\text{ROC 2:} \quad 0.2 < |z| < 0.6 \\
\text{ROC 3:} \quad |z| < 0.2
\]
System Function for Interconnections of LTI Systems

\[ H(z) = H_1(z) \cdot H_2(z) \]

\[ H(z) = H_1(z) + H_2(z) \]
System Function for Interconnections of LTI Systems

\[ \frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)} \]
Block Diagram of Causal LTI systems described by Difference Equations and Rational System Functions.

Example 10.28 A Causal LTI system with system function:

\[ H(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} \]

\[ \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{4} z^{-1}} \]

\[ Y(z) \{1 - \frac{1}{4} z^{-1}\} = X(z) \]

Taking the inverse z-transform of the above equation,

\[ y[n] - \frac{1}{4} y[n-1] = x[n] \quad \text{or} \quad y[n] = x[n] + \frac{1}{4} y[n-1] \]